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Price dynamics from a simple multiplicative random process model

Stylized facts and beyond?

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Abstract. The existence of stylized facts suggests that there might be ‘universal’ mechanism which drives price evolution on financial markets in general. Based on empirical estimates of 10 major indices, we propose a stylized model of endogenous price formation on an aggregate level whose key issue is that price evolution is driven by the ‘market’s’ expectations about future growth rates of investment. The model is a multiplicative random process with a stochastic, state-dependent growth rate which establishes a negative feedback component in the price dynamics which admits some far reaching formal analysis. Generated return trails exhibit statistical properties such as ‘volatility clustering’, multi scaling, and a non-Gaussian distribution which is in quantitative in agreement with stylized facts from empirical asset returns. Additionally non-equilibrium entropies are also considered. These results suggests that the structure of the model mimicks a mechanism which is essential in driving price dynamics of financial markets in general.

PACS. 89.65.Gh Economics; econophysics, financial markets, business and management

A financial market is constituted by a large number of financial agents which interact by trading assets. During trade asset prices are determined due to aggregate demand, mediated by the trading mechanism of the respective stock exchange. Therefore prices have to be regarded as *macro-observables*. Statistical Physics suggests that macro-observables, when being typical properties of the system, are independent from micro realizations of the complex system and therefore are common to (almost) all its realizations [1]. Consequently prices should be independent from individual trading decisions. Apart from practical implementation problems, this seriously asked whether for modeling these macro observables it is necessary to write down individual motions of equations. Consequently, when describing prices, we neglect any so-called micro-founded considerations of individual decision making.

The statistical properties that are invariant under the choice of a particular realization of the system, are called ‘stylized facts’, for a survey see [2] and also the monographs [3,4] as well as the references in them. Stylized facts can be regarded as constraints for any modeling attempt. During the last years a number of models have been proposed which are able to produce selected stylized facts. Some of them are based on the mathematical assumptions such as stability [5] or multifractality [6,7], while others are mostly descriptive in that they consist essentially in postulating that prices follow particular stochastic processes which generate certain distributions such as Levy-processes [8] or hyperbolic processes [9–12]. Within the zoo of models, the family of models of interacting agents also have to be mentioned, see [13] for a detailed review. However, data do not admit a unique choice among these models. In other words, stylized facts are not strong enough to single out a unique model of a financial market. Simplicity of the model and the possibility for far-reaching analytical tractability might be a further criterion.

Our viewpoint is that the existence of stylized facts suggests that price trails might be considered as realizations of a more general random/complex system, called ‘The Financial Market’. Stylized facts, then, are statistical properties which are typical for *realizations* of the financial market. Given this view, the question follows whether there might exist a simple and economically plausible mechanism which drives a financial market in general.

The modeling approach in this note is to start by proposing a zero-order model for the dynamics on a macro level, which is as simple as possible, then to investigate its properties in some depth and to compare these properties with those of real data, to recognize the differences between this zero-order model, and then to add more structure to it to bring its properties closer to empirical estimates. Along this way, we hope to successively obtain more knowledge about fundamental features of a

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Table 1. For each index, we considered daily data from 01/01/1990 to 12/31/2004 provided by Thompson Datastream. The NIKKEI is from 10/30/1986 to 12/30/2005.

1	France	CAC 40
2	Germany	DAX 30
3	Hong Kong	Hang Seng
4	Japan	NIKKEI 225
5	Switzerland	Swiss Market Index
6	Switzerland	Swiss Performance Index
7	United Kingdom	FTSE 100
8	United States	Dow Jones IndAvg
9	United States	Nasdaq 100
10	United States	S&P500

mechanism driving price dynamics in financial markets in general.

1 On the empirical basis

"It may well be true that a set of assumptions is reasonable from a logical standpoint. But this does not prove that it corresponds to nature'. Your are right, dear skeptic. Experience alone can decide on truth"

A. Einstein

In this section, we sketch the empirical basis we are concerned with. We consider the dynamics of prices $(X_t)_{t\geq 0}$ and the statistical properties of price changes. The approximation of relative price changes by the corresponding logarithmic returns

$$\frac{X_{t+1}-X_t}{X_t} \approx \ln\frac{X_{t+1}}{X_t} \; =: \; \tilde{Z}_t$$

is reasonable if $\|X_{t+1}-X_t\|$ is sufficiently small. This poses some restrictions on the smoothness of the market and/or on the time scale to be considered. For comparison of return trails from different markets, we standardize these trails according to

$$Z_t = \frac{\tilde{Z}_t - \mu(\tilde{Z}_t}{\sigma(\tilde{Z})}$$

where $\mu(\tilde{Z})$ is the mean of $(\tilde{Z}_t)$ and $\sigma(\tilde{Z})$ the standard deviation. By construction $\mu(Z)=0$ and $\sigma(Z)=1$ for all trails (Z_t). We considered the 10 major indices displayed in Table 1.

In this note we consider three (classes) of major stylized facts:

– 'volatility clustering': the existence of long-lasting auto correlations in absolute returns;
– multi scaling behavior: the non-linear spectrum of local Hurst exponents
– distribution of relative price changes: its non-Gaussian character and skewness.

These statistical properties are qualitative the same for all indices. Therefore we can restrict ourselves to presenting only present one index, which is the NIKKEI 225, in the following.

Figure 1 shows these stylized facts for the NIKKEI 225 index, daily data from 10/30/1986 to 12/30/2005. The upper left picture shows the return trail of the NIKKEI 225. It clearly exhibits some degree of '*volatility clustering*'. The upper right picture displays the slow, polynomial, decay of the autocorrelation of squared absolute returns, which is taken as a quantitative measure of 'volatility clustering'. Due to the existence of long lasting memory effects in the system, the *singularity spectrum* is non-linear. Finally, the *distribution of returns* is shown in the lower right picture. The deviation from the inscribed Gaussian curve (dashed black parabola) clearly shows that the distribution is not Gaussian and slightly skewed, while tails are slightly fatter than those of the exponential distribution. Some remarks on the mentioned statistical properties may be in place here.

Concerning volatility clustering it might be important to recall that volatility is not a direct observable [2]: while the shape of the return trail suggests a decomposition of returns in a product of some pure noise ϵ_t and a conditioned 'volatility factor' σ_t, i.e. $Z_t = \sigma_t\,\epsilon_t$, this decomposition is only satisfied on the basis of some pre-given model like GARCH. Particularly the volatility factor σ_t is not model free. Instead we consider the decay of autocorrelations of integer-powers of log-returns. As initially pointed out by Ding and Granger, the decay is slow, i.e. polynomial

$$C_a(\tau) := cov(|Z_{t+\tau}|^a, |Z_t|^a) \;\sim\; \tau^{-\gamma_a}, \qquad a = 1,2,\ldots$$

where $\gamma_a < \gamma_{a'}$ whenever $a < a'$. The polynomial decay of squared absolute returns $C_2(\tau)$ is taken as a measure for volatility clustering. In a double-logarithmic plot of $C_\alpha(\tau)$ versus α, we expect to see a line for large τ.

The spectrum of local Hurst exponents serves as a measure of the regularity of the time-trails. While the singularity spectrum $\tau(q)$ of a fractional diffusion process is a linear function of the moments q, the empirical spectrum is clearly non-linear. Note that a system generating a time series exhibiting multi-fractal behavior is not necessarily multi-fractal itself. Aside from simple statistical reasons such as finite coarse-graining and the finiteness of the sample size, Bouchaud et al. pointed out that long lasting effects as induced by volatility clustering can automatically lead to this multi-scaling behavior. As shown in [14], short time series of mono-fractal processes such as simple random walks may exhibit a nonlinear Hurst spectrum, while so-called *apparent multi-fractality* is also known from other systems, see [15], including multiplicative random walk as well [16]. This particularly means that a non-trivial singularity spectrum does not necessarily imply that the financial market is truly multi-fractal. Models of a financial market as a true multi-fractal system have been proposed quite recently [6,7,17]. For review and an extended literature survey about multi-fractality in finance see [18,19]. For a critical review also see [20]. However, these findings may help to understand why the singularity spectrum of assets *typically* is non-linear and

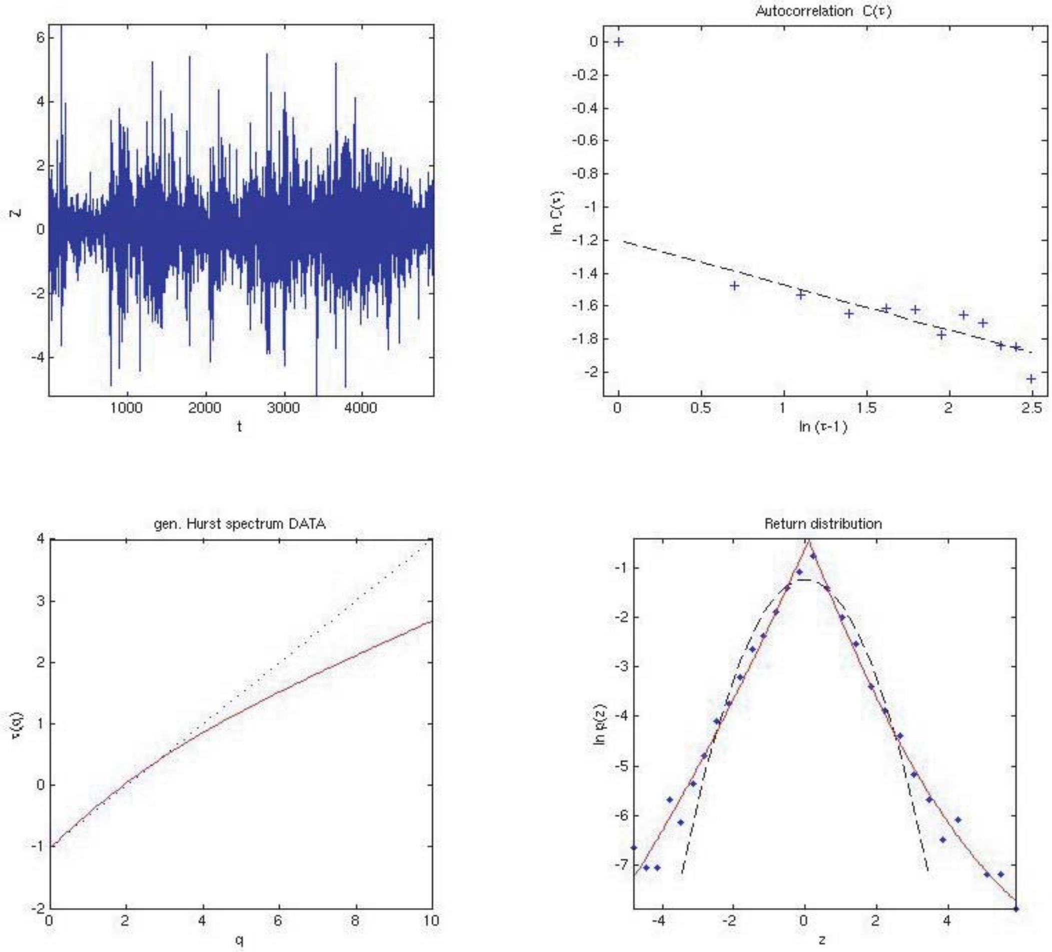

Fig. 1. Stylized facts of empirical asset returns: daily data of the NIKKEI 225 from 10/30/1986 to 12/30/2005.

moreover seems to be equal for all assets. However, the singularity spectrum of the NIKKEI return trail is non-linear.

Finally, the DISTRIBUTION OF LOG-RETURNS is non-Gaussian, in that large returns have a significantly higher probability to occur than to be expected if the distribution would the a Gaussian. Careful investigations have shown that empirical distributions are well fitted in the center by a Levy distribution, while their tails are typically less heavy than those from a Levý distribution. Alternative distributions have been proposed in the past, based on various assumptions such as stability and multi-fractality, as well as based on analogies with other processes including turbulence or transport.

1.1 A first suggestion from data?

Often the inspection of data gives a first orientation for later modeling. In this section we investigate our data which are daily returns over a 15-years period. Quantile-Quantile plots provide a good descriptive method to judge about whether two sets of data come from the same distribution. A line indicates that data in both sets are very likely to come from the same distribution. As an example, we consider the NIKKEI 225. Figure 2 shows the QQ-plot of the empirical time series of log returns Z with respect to an exponential distribution. The left column considers positive log-returns $Z_+ := Z_{\geq 0}$, while the right column is for negative log-returns $Z_- := Z_{\leq 0}$. The first row displays the QQ-plots of $Z_\pm$ which respect to the exponential, while the second row displays a QQ-plots for $\ln |Z_\pm|$ with respect to the logarithm of the exponential. The scheme see in Figure 2 thus is:

Z_+	Z_-
$\ln Z_+$	$\ln \lvert Z_- \rvert$

The QQ plots of all indices considered show the same pattern: while the middle part of each QQ-plot is linear, systematic deviations from the straight line in the QQ-plots occur for either small returns or large returns. Particularly, deviations from the diagonal for the QQ-plot of $Z_\pm$ are seen for high quantiles, while deviations form the line in the QQ-plot of $\ln |Z_\pm|$ exists for small quantiles. This means that empirical returns deviate from being exponentially distributed for large returns, see the first row, and for very small returns, see the second row. This agrees with the observations in the respective pdf's, where we

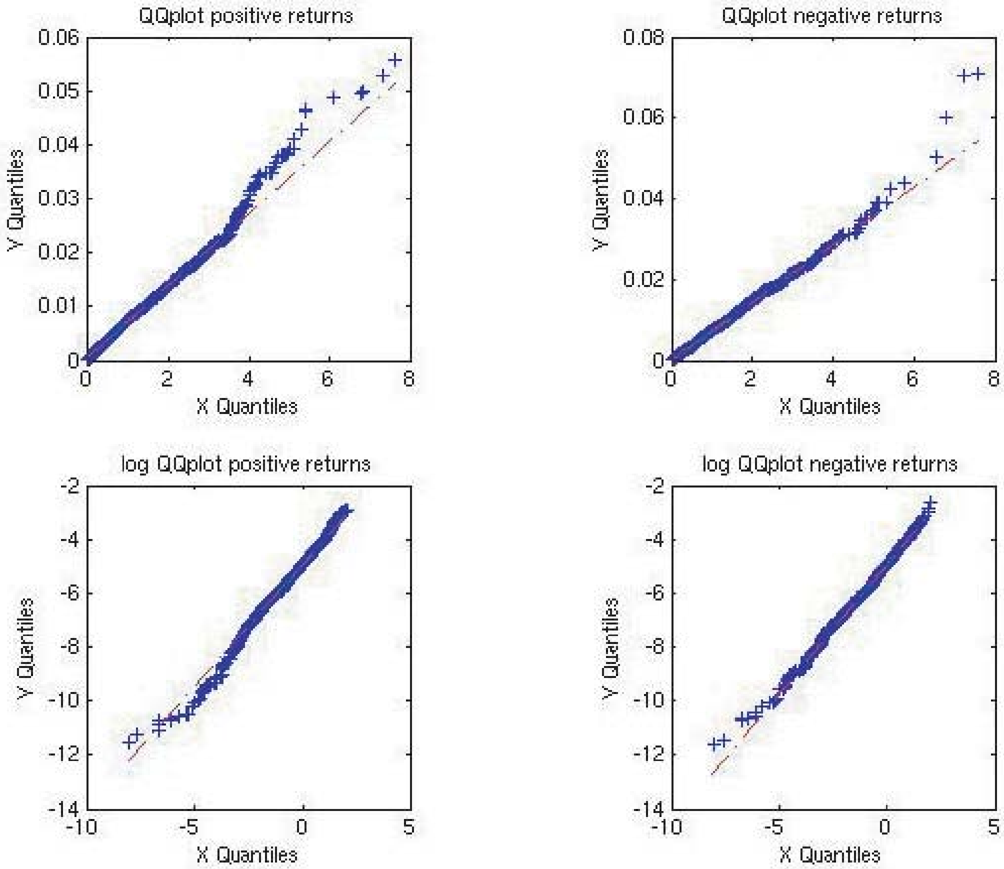

Fig. 2. QQ-plots of the NIKKEI with respect to the exponential distribution.

see that typically the empirical pfd has less mass in 0 than the Laplacian, while it's tails are usually fatter than those from the Laplacian.

The goodness of this eye-bowling impression can be made more quantitative as follows. We compare two distributions with respect to their goodness to fit empirical data. One is the exponential with parameter β, while the other one is the Weibull with parameters (a, μ)

$$f_Z^{exp}(z) = \frac{1}{\mu}\, e^{-\frac{z}{\mu}}$$
$$f_Z^{WB}(z) = \frac{m}{z}\left(\frac{z}{a}\right)^m\, e^{\left(-\frac{z}{a}\right)^m}.$$

The exponential distribution is obtains from the Weibull in the case $m = 1$, while in the limit $m \to 0$, $f_Z(z)$ approximates the Pareto distribution arbitrarily well, see [21]. Fits of the distributions for positive and negative returns due to these distributions therefore give respective parameters

$$\boldsymbol{\mu} = \left(\mu^+, \mu^-\right)$$
$$\mathbf{m} = \left(a^+, m^+;\ a^-, m^-\right).$$

Therefore estimating the parameters $\boldsymbol{\mu}$ and $\boldsymbol{m}$ is of special interest. In the following we summarize log likely fits of positive and negative returns, respectively, to the exponential distribution and the Weibull distribution, giving parameters $\mu^{\pm}$ and $(a^{\pm}, m^{\pm})$ respectively.

Distributions are close to being symmetric $\mu_+ \approx \mu_-$, see Tables 2 and 3. Furthermore $0 \ll m^{\pm} \approx 1$, i.e. their distributions are close to an exponential distribution. This finding is also supported by considering the respective entropic distances between the empirical distribution and the exponential and Weibull distribution respectively as formalized by the Akaike's Information Criterion

$$AIC = -2\ \log\left(\mathcal{L}(\hat{\theta}|x)\right) + 2\,K,$$

where K is the number of parameters, here $K = 1$ for the exponential and $K = 2$ for the Weibull distribution. The term $2K$ can be considered as a penalty for introducing additional parameters. $\mathcal{L}(\hat{\theta}|x)$ is the maximum log-likelihood of the parameter θ given the data x. AIC is in particular useful for nested model such as the exponential and the Weibull in this case. However, we considered the ratio of the AIC's of both models

$$\text{ratio} \;=\; \frac{\mathcal{L}(\text{exponential}) + 1}{\mathcal{L}(\text{Weibull}) + 2}.$$

Table 2. Parameter estimation for negative returns of the indices considered.

INDEX	# $Z < 0$	μ^-	a^-	m^-	ratio
DAX 30	1806	0.6985	0.6613	0.8885	0.6671
SWISS SMI EXP	1066	0.7021	0.6762	0.9209	0.6673
FRANCE CAC 40	1849	0.7054	0.6869	0.9399	0.6671
FTSE 100	1840	0.7074	0.6997	0.9742	0.6671
SWISS SPI EXTRA	955	0.7342	0.6891	0.8820	0.6673
DOW JONES	1793	0.6891	0.6713	0.9437	0.6671
Hang Seng	1808	0.6393	0.6000	0.8815	0.6671
NASDAQ 100	1745	0.7077	0.6887	0.9405	0.6671
NIKKEI 500	1864	0.7500	0.7698	1.0688	0.6671
S&P 500	1799	0.6869	0.6663	0.9353	0.6671

Table 3. Parameter estimation for positive returns of the indices considered.

INDEX	# $Z > 0$	μ^+	a^+	m^+	ratio
DAX 30	1964	0.7057	0.7280	1.0840	0.6671
SWISS SMI EXP	1190	0.7003	0.7186	1.0683	0.6673
FRANCE CAC 40	1925	0.7454	0.7786	1.1285	0.6671
FTSE 100	1948	0.7339	0.7638	1.1129	0.6671
SWISS SPI EXTRA	1283	0.6634	0.6984	1.1615	0.6673
DOW JONES	1979	0.7223	0.7478	1.0948	0.6671
Hang Seng	1904	0.7038	0.7105	1.0229	0.6671
NASDAQ 100	2033	0.6933	0.7070	1.0485	0.6671
NIKKEI 500	1825	0.6592	0.6173	0.8699	0.6670
S&P 500	1983	0.7207	0.7373	1.0589	0.6671

As seen in Tables 2 and 3, the ratio is close to 2/3, which indicates that the distribution of log returns can quite well be approximated by an exponential. This evidence will be important in the following.

In summary, investigations of the distribution of the NIKKEI suggests that the distribution of logarithmic returns is close to an exponential, while there are systematic deviations in the center, i.e. for small returns, and also for large returns. Particularly, small returns are less often than expected from the exponential distribution, while large returns are more often. Thus, if taking the exponential distribution as a zero-order distribution for daily returns, one has to consider these deviations in a first-order approximation.

2 The basic model

People go to the financial market to make their money work. They do so by investing their money into *promising* assets. At some time t, the agent has capital m_t and decides to invest a portion λ_t of it in asset A. His request in this asset may depend on its current price X_t; we therefore write $\lambda_t = \lambda(X_t)$. For $\lambda_t m_t$, he can buy $|A_t| = \lambda_t m_t / X_t$ units of this asset for its price X_t. One unit of this asset has an uncertain value $\tilde{D}_{t+1}$ some time later. Hence in $t+1$, the money m_t invested in asset A has value $M_{t+1} = \tilde{D}_{t+1}\lambda_t / X_t \, m_t$, and the money m_t invested has become 'more' valuable by a factor

$$\gamma_{t+1} = \frac{\tilde{D}_{t+1}\lambda_t}{X_t}. \qquad (1)$$

While the growth rate γ_{t+1} is uncertain at time t, the 'market' builds an expectation about the future growth rate at time t.

$$\Gamma_{t+1} = \varphi_t(\gamma_{t+1}), \qquad (2)$$

where φ_t is assumed to be some strictly increasing function in its argument. An asset is *promising* if 'the market' expects that its value will increase, i.e. if $\Gamma_{t+1} > 1$. It is reasonable to assume that the demand in an asset is higher the more promising it is. It then follows from equation (2), that if the price is high, the probability that Γ_{t+1} will be larger than 1 is small and hence excess demand is likely to be be small, while, on the other hand, if the price is low, the probability for a growth rate larger than 1 is high, so that excess demand in an asset with low price is high. Since, according to standard economic thinking, prices follow the excess demand, price evolution is intimately linked to the market expectations about the future growth rate of an asset.

Accordingly the price process is driven by the aggregate expectation of 'the market' about future growth rates of values, while the above mechanism establishes a strong multiplicative stochastic feedback component in the dynamics of an expectation driven market. The above considerations then gives rise to a simple dynamical system for price evolution, see equations (3) and (9). For further details see [22]

2.1 Basic assumptions

Our model will be based on the following general assumptions, i.e. it is restricted to the following settings. We consider a market with only one asset, whose future value is described by some non-negative factor $\tilde{D}$. This factor summarizes all exogenous influences that drive the future value of the asset. Since the future value is uncertain, we assume that it is distributed according some distribution. This distribution is assumed to be stationary in time. Since the very nature of this value factor is left open, we only assume that it it takes finite values. Then, in order to suppose a minimum of further assumptions, we assume that $\tilde{D}$ is uniformly distributed in some non-negative, finite interval. These throughout assumptions are summarized in assumption (A).

$$[\,0,\ t\,)\ =\ [\,0,\ 1)\ \star [\,1,\ 2)\ \star \ldots\ \star [\,t-1,\ t).$$

The next specification concerns the structure of the trading periods. For the sake of simplicity let $X_{[t}$ denote the 'opening price' in the period $[t, t+1)$ and $X_{t+1)}$ its closing price. Consecutive trading periods are called 'independent' if $X_{t+1)}$ is independent of $X_{[t+1}$, i.e. the opening price of period $[t+1, t+2)$ is independent of the closing price of the former period $[t, t+1)$. If, in the other extreme $X_{t+1)} = X_{[t+1}$, then the trading periods are said to be dependent of each other. Reality tells us that that the truth is somewhere in between. However, these two extremes are considered as well, see Assumption (B).

The third class of assumptions concerns the 'market' expectation about future growth rates of the asset. The assumptions on this function φ_t made are that $\varphi_t(0) = 0$, while it is strictly increasing. We will deal with some basic variations. One is that this function is a power function. We also consider the case that this function is a logarithmic function. In this case market expectations are also concave but unbounded. Finally we also consider this function to be a negative exponential, which is concave and bounded. The following distinction is more important: The market expectation can be constant, i.e. $\varphi_t = \varphi$, or it can fluctuate around some φ_0. This is the content of Assumption (C).

Here is a list of the assumption dealt with in this note. In the following section we will explicitly state which particular assumptions are make at various levels.

(A) *The structure of the market*: one asset with stationary and uniformly distributed uncertain value.
(B) *The structure of trading periods*: consecutive periods are **(1)** independent from each other or **(2)** dependent.
(C) *The 'market expectation'*: believes about the future growth rate can **(1)** be constant or **(2)** fluctuate over time.

The *zero-order model* uses assumptions (A, B_1, C_1), while the *first order model* uses assumptions (A, B_2, C_2).

3 The 0-order model

In this lowest order approximation the structure of the financial market is simple: consecutive trading periods are independent from each other while the market expectation is the same in each period. The assumptions therefore are

(A) *The structure of the market: one asset with stationary and uniformly distributed uncertain value.*
(B_1) *The structure of trading periods: consecutive periods are independent from each other.*
(C_1) *The 'market expectation': believes about the future growth rate are constant over time.*

Let X_t be the opening price and X'_t the closing price of period $[t, t+1)$. Then within $[t, t+1)$ the price X_t evolves to X'_t due to a multiplicative process given by

$$X'_t = \Gamma_{t+1}\, X_t \tag{3}$$

according to the expected growth rate of the investor given by equation (2). In this setting the distribution over a sequence of periods is the average over the outcomes of all periods $[t, t+1)$, while all periods are indistinguishable. We therefore omit the subscripts in the following.

Recall that the value factor $\tilde{D}$ is assumed to be uniformly distributed in a finite interval $[0, d]$. We define

$$\alpha(X) := d\ \frac{\lambda(X)}{X}.$$

Due to these assumptions the growth rate Γ yields

$$\Gamma = \varphi_t\Big(\delta\,\alpha(X)\Big) \qquad \delta \sim \mathcal{U}(0,1). \tag{4}$$

Given the opening price x, the conditioned probability that the price relative is larger than some $r \geq 0$ is given by

$$\overline{F}_R(r|x) = P[R > r|x]\ =\ 1 - \frac{1}{\alpha(x)}\ \varphi_t^{-1}(r).$$

For un-conditioning we average over all possible initial prices x in the range $0 \leq x \leq \chi$ to obtain the averaged tail cdf of price relatives R

$$\overline{F}_R(r) = \chi - \varphi^{-1}(r)\ \int_0^{\chi} \frac{1}{\alpha(x)} dx$$

where χ is determined by the condition $\frac{1}{\alpha(\chi)}\varphi^{-1}(r) = 1$.

3.1 Effect of price depending demand

Under the assumption that the demand depends on weakly on the current price, i.e. $\lambda(x)\ =\ \lambda\,(1 + \epsilon\, x)$, where ϵ is small, we obtain from $\chi \approx \frac{\lambda}{\varphi^{-1}(r)} + \epsilon\frac{\lambda^2}{\varphi^{-1}(r)^2} + \mathcal{O}(\epsilon^2)$ for the unconditioned tail distribution

$$\overline{F}_R(r) = \frac{\lambda}{2}\,\frac{1}{\varphi^{-1}(r)}\,\left(1 + \epsilon\,\frac{2\,\lambda}{3}\,\frac{1}{\varphi^{-1}(r)}\ \right)\ +\ \mathcal{O}(\epsilon^2). \tag{5}$$

Defining $\phi(r) := \varphi^{-1}(r)$, the unconditioned pdf of price relatives yields

$$f_R(r) = \frac{\lambda}{2}\,\frac{\phi'(r)}{\phi(r)^2}\,\left[1\ +\ \epsilon\ \frac{4}{3}\ \frac{\lambda}{\phi(r)}\right]\ +\ \mathcal{O}(\epsilon^2).$$

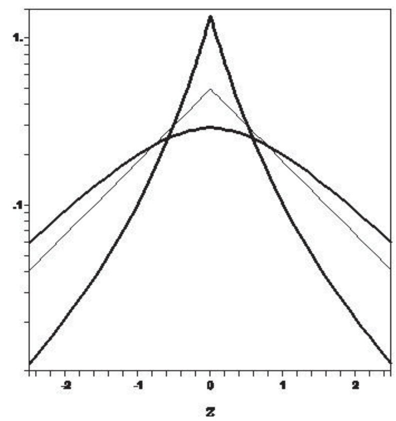

Fig. 3. For $\epsilon > 0$, $f_Z^{(\epsilon)}$ is convex, while it is a Laplacian for $\epsilon = 0$, and for $\epsilon < 0$ it is concave.

Since φ is non-negative and increasing, so is ϕ. We can therefore define a function $\Phi_\epsilon(r)$ so that $f_R^\epsilon(r) = \lambda\, e^{-\Phi_\epsilon(r)}/2$. The average tail distribution of log-returns $Z = \ln\, R$ yields

$$f_Z^\epsilon(z) = \frac{\lambda}{2\mu_\epsilon}\; \exp\left\{ z - \Phi_\epsilon\left(e^z\right) \right\}. \tag{6}$$

with normalization constant μ_ϵ.

Observation 1 (Effect of price depending demand.) *Let price dependence of demand be*

$$\lambda_t = \lambda(1 + \epsilon X_t), \qquad \epsilon < 0,$$

Note that $\Phi_0(e^z) = -\ln \frac{\phi'(e^z)}{\phi(e^z)^2}$, *while* $\frac{\Phi_\epsilon(e^z)}{\Phi_0(e^z)} \to 1$ *for* $z \to \infty$. *Consequently the correction term vanishes for large* z. *For large* $|z|$ *the distribution is independent of* ϵ*! For small* $|z|$, Φ_ϵ *gives a positive correction for* $\epsilon > 0$ *and a negative correction for* $\epsilon < 0$.

Example 1 *For illustration we chose* $\varphi(r) = r^\mu$, $\mu > 0$. *In this case* $f_Z^0(z) = e^{\frac{|z|}{\mu}}/2\mu$ *is a Laplacian with intensity* $1/\mu$. *In a semi-logarithmic plot, its graph is a 'hard tent'. If the demand function* λ_t *weakly varies with the price level, the 'hard tent' becomes deformed in the central part, i.e. for small returns. If demand decreases in prices,* $\epsilon < 0$, *the distribution becomes a soft tent, i.e.* $f^{(\epsilon)}$ *is differentiable in* $z = 0$, *while if the demand increases with the price level, i.e.* $\epsilon > 0$, *the distribution is convex, see Figure 3.*

3.2 Various simple 'market' expectations

Recall that the function φ_t represents the aggregate expectations of future growth rates in the market in period $[t, t+1)$. Let us assume that $\varphi_t = \varphi_0$ for all periods. The question then is, which expectation function φ_0 is reasonable? In principle equation (6) allows one to construct an expectation function so that the corresponding distribution fulfills given properties. Since φ_0 is not observable, the choice of this function contains some degree of arbitrariness. From an aggregation point of view this function would have to be estimated from aggregating the expectations of individual traders on the market. Thereby arbitrariness is shifted down to the level of individuals. On the other hand, since we want to propose a simple model of price dynamics on the aggregate level and thus without referring to any individual level, we stay with simple functional forms which are commonly used in economic modeling as well. Recall that the basic assumption on an expectation function φ_0 is that $\varphi(0) = 0$ while it is strictly increasing in its argument. Three classes of functions are the following:

1. the negative exponential $\varphi_0(x) = \nu\,(1 - e^{-\mu x})$;
2. a logarithmic function $\varphi_0(x) = \ln(1 + \mu x)$;
3. a power function $\varphi_0(x) = x^\mu$.

In all cases $\alpha, \beta > 0$. The negative exponential differs from the other functions in that it is bounded by α, while the others are unbounded. This shows off also in the distributions generated. A straight forward calculation shows that these functions generate the following return distributions:

$$f_Z^0(z) \sim \frac{1}{4}\ln\frac{\nu}{\nu-1}\frac{e^{|z|}}{\left(\nu - e^{|z|}\right)}\ln\left(1 - \frac{1}{\nu}\, e^{|z|}\right)^{-2},\ |z| \le \ln\nu$$

$$\to \frac{1}{2}e^{-|z|} \quad \text{for } \nu \to \infty$$

$$f_Z^0(z) \sim \frac{e-1}{2}\, e^{|z|}\, \frac{e^{e^{|z|}}}{\left(1 - e^{e^{|z|}}\right)^2} \;\to\; \frac{e-1}{2}\, e^{|z| - e^{|z|}}$$

$$\text{for } |z| \to \infty$$

$$f_Z^0(z) \sim \frac{1}{2\,\mu}\, e^{-\frac{1}{\mu}\,|z|}.$$

Obviously the distribution for the negative exponential is defined only for sufficiently small returns $|z| \le \ln\nu$, while it has a singularity in $|z| = \ln\nu$. In the limit $\nu \to \infty$, tails decay exponentially. The distribution of the logarithmic expectation is parameter free, i.e. independent of μ and, asymptotically, decays faster than exponentially. Finally, the power function generates a symmetric Laplacian with intensity $1/\mu$, i.e. its tails are semi-fat tailed.

As already discussed in the previous section, the exponential distribution does a good job. The negative exponential has the property that the behavior in the tails is independent from any parameter, while the distribution corresponding to the log-function decays faster than the exponential, while data clearly show that the decay is faster. The goodness of the exponential distribution function is also justified by the following finding.

3.3 The effect of fluctuating market expectations

From Section 2 it is clear that the tails of the empirical return distributions are more heavy than those of

Table 4. The range of standardized returns of indices indicates typically is within $[-10, +10]$. The corresponding distributions are shown left.

INDEX	min Z	max Z
DAX 30	–6.8099	5.1869
SWISS SMI EXP	–5.8302	5.7350
FRANCE CAC 40	–5.7229	5.1957
FTSE 100	–5.7071	5.6899
SWISS SPI EXTRA	–7.8588	4.3435
DOW JONES	–7.4663	6.1007
Hang Seng	–9.2021	10.7154
NASDAQ 100	–5.3480	8.7963
NIKKEI 500	–5.7149	8.2194
S&P 500	–22.1810	8.4101

the exponential distribution. So, what is missing in our model? Recall that we assumed that the market expectation is constant over all periods. On the other hand it is more realistic to assume that market expectations are not constant rather than fluctuate. The reason for time varying market expectations are manifold. In the following we stay with the setting of independent trading periods but now let the market expectation φ_t fluctuate over periods around some 'mean' which is denoted by φ_0.

In this section we analyze our model under the following assumptions:

A *The structure of the market: one asset with stationary and uniformly distributed uncertain value.*
B_1 *The structure of trading periods: consecutive periods are independent from each other.*
C_2 *The 'market expectation': believes about the future growth rate fluctuate over time.*

In the case of the power function, we therefore have

$$\varphi_t(\lambda) \sim \lambda^{\mu_t}$$

where μ_t is a random variable taking values in some narrow range around μ, i.e.

$$\mu - \epsilon \le \mu_t \le \mu + \epsilon, \qquad 0 \le \epsilon \le \mu.$$

Assuming $\mu_t \sim \mathcal{U}([\mu-\epsilon, \mu+\epsilon])$ we average the distribution $f_Z^{\mu}(z)$ over the interval $[\mu-\epsilon, \mu+\epsilon]$ to obtain

$$f_Z(z) \;=\; \frac{1}{4\epsilon}\left[\Gamma\left(0, \frac{|z|}{\mu+\epsilon}\right) - \Gamma\left(0, \frac{|z|}{\mu-\epsilon}\right)\right]. \qquad (7)$$

Expanding this expression for small ϵ yields

$$\tilde{f}_Z(z) \;=\; \frac{1}{2\mu}\, e^{-\frac{\|z\|}{\mu}} \left[1 + \epsilon^2\, c(|z|) \;+\; \mathcal{O}(\epsilon^3)\right] \qquad (8)$$

where the correction term is

$$c(|z|) = \frac{1}{3\,\mu^3}\left[\mu - 2\,|z| + \frac{1}{2\mu}\,|z|^2\right]$$

which vanishes for $|z| \to 0$. Note that $\int_{\mathbb{R}} \tilde{f}_Z(z)\, dz = 1$.

Due to the form of the correction term, fluctuations of market expectations in our model generate semi-heavy tails in the distribution, i.e. tails decay exponentially. The effect of market fluctuations therefore decreases the large μ is.

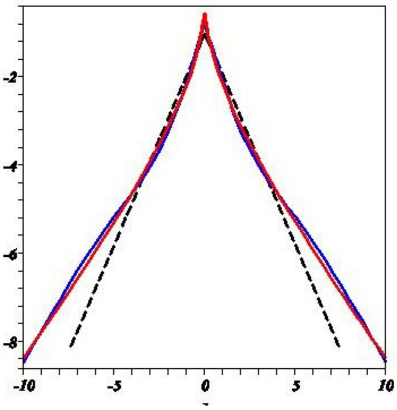

Fig. 4. Return distribution in our model with fluctuating liquidity parameter μ. The dashed line displays the Laplacian, the red graph is the exact distribution $f_Z(z)$, while the blue one corresponds to our approximation $\tilde{f}_Z(z)$.

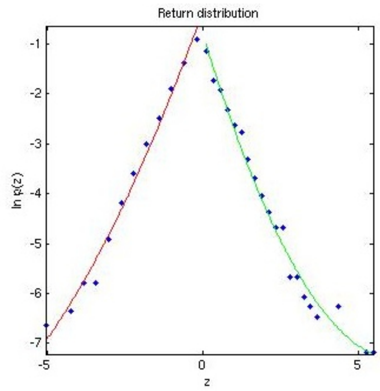

Fig. 5. Fitting the return distribution of the Nikkei.

Proposition 1 (Effect of fluctuations.) *Fluctuations in the aggregate market expectation affect the distribution of large returns. Particularly, tails are fatter the stronger fluctuations are.*

According to equation (3.3) the shape of the distribution in a semi logarithmic plot is described by

$$\ln \tilde{f}_Z(z) \;=\; c_0 + c_1\,|z| \;+ c_2\,|z|^2,$$

where $c_0 = -\ln 2\mu + \epsilon^2/(3\mu^2)$, $c_1 = -\left(1 + 2\epsilon^2/(3\mu^2)\right)/\mu$, and $c_3 = 1/(2\mu^2)\,\epsilon^2/(3\mu^2)$. Fitting to empirical data gives a satisfactory result.

4 Dynamics from depending trading periods

So far we considered the situation that successive trading periods were independent from each other. Particularly opening prices are independent from former closing prices. This certainly contradicts intuition. However, in this setting the resulting distribution is symmetric with respect

Table 5. The *skewness* γ, and the *kurtosis* κ of the standardized daily returns of the indices considered.

INDEX	γ	κ
DAX 30	–0.2364	6.9411
SWISS SMI EXP	–0.1722	6.9098
FRANCE CAC 40	–0.0895	5.8251
FTSE 100	–0.0934	6.1410
SWISS SPI EXTRA	–1.0403	7.9648
DOW JONES	–0.2295	7.6912
Hang Seng	–0.0308	13.0648
NASDAQ 100	0.1240	7.5741
NIKKEI 225	–0.1257	10.4664
S&P 500	–1.7449	42.9679

to 0, i.e. the probability for having a return of z is the same as having a return of value $-z$. This is in contradiction with empirical findings. In fact return distributions are slightly skewed. This fact can not be explained in our formal model version.

The following table displays the list of indices and the non-trivial moments of the standardized returns Z_t. By construction the first two moments are $\mu(Z) = 0$ and $\sigma(Z) = 1$, while the skewness γ and the kurtosis κ yield:

Let us take seriously that opening prices are not independent from the former closing prices! The set of assumptions now is the following:

A *The structure of the market: one asset with stationary and uniformly distributed uncertain value.*

B_2 *The structure of trading periods: consecutive periods are dependent.*

C_1 *The 'market expectation': believes about the future growth rate are constant.*

The assumption that related closing and opening prices are independent is unrealistic, of course. The other extreme is that related closing and opening prices are identical in the sense that

$$X_{t)} \;=\; X_{[t}.$$

In this case the evolution of prices is described by a simple dynamical system given by

$$X_{t+1} \;=\; \mathit{\Gamma}_{t+1}\, X_t, \qquad t \leq 0. \tag{9}$$

Dynamics leave basic features of the distribution in the stationary case unchanged, while it implies a qualitatively new one feature: while the distribution in the stationary case is symmetric, the distribution is asymmetric when considering dynamics. This shown in the following.

4.1 Dynamics and the asymmetry of the distribution

Since the power function $\varphi(x) = x^\mu$ already did a good job, the following analysis is done for this function. In the following we show that the resulting distribution is symmetric for $\mu = 1$, while it is positively skewed for $\mu > 1$ and negatively skewed for $\mu < 1$.

This argumentation can be made precise by considering price evolution as a process, i.e. assuming that $X_{t+1} = X'_t$, in which consecutive prices are related by

$$X_{t+1} \;=\; \delta_t^{\mu} \quad X_t^{1-\mu}. \tag{10}$$

Logarithmic prices $\zeta_t = \ln X_t$ then satisfy the difference equation $\zeta_t := \mu \ln \delta_t + (1-\mu)\ \zeta_{t-1}$, whose generating function — for $\zeta_0 = 0$ — yields

$$F_\mu(s) \;=\; \mu \sum_{t\geq 1} \frac{\ln \mu\, s^t}{1 - s + \mu s}. \tag{11}$$

The coefficients $c_\mu(t)$ of its Taylor expansion in $s = 0$ obey $c_\mu(t) = \zeta_t$ and hence $Z_\mu(t) = \zeta_t - \zeta_{t-1} = \ln \frac{X_t}{X_{t-1}}$ is obtained from

$$Z_\mu(t) = c_\mu(t) - c_\mu(t-1). \tag{12}$$

If $\mu = 1 - a$, $|a| \ll 1$, expansion of $Z_\mu(t)$ in equation (12) around $\mu = 1$ up to first order in μ then gives

$$Z_\mu(t) = \ln\left(\delta_t^{1-a}\ \delta_{t-1}^{2a-1}\ \delta_{t-2}^{-a}\right) \;+\; \mathcal{O}(a^2) \tag{13}$$

Z_μ is the sum of the following random variables Y_j derived from $\delta_{t-j} \sim \mathcal{U}(0,1)$, $j = 0, 1, 2$ with probabilities $f_{Y_j}(z)$ respectively

$$\begin{array}{ll} Y_0 = (1-a)\ \ln \delta_t, & f_{Y_0}(z) = c_0\ e^{\frac{z}{1-a}}\ I_{(-\infty,0)} \\ Y_1 = (2a-1)\ \ln \delta_{t-1}, & f_{Y_1}(z) = c_1\ e^{\frac{z}{2a-1}}\ I_{(0,\infty)} \\ Y_2 = -a\ \ln \delta_{t-2}, & f_{Y_2}(z) = c_2\ e^{\frac{z}{-a}}\ I_{(0,\infty)} \end{array}$$

where normalization constants yield $c_0 = 1/(1-a), c_1 = 1/(1-2a), c_2 = 1/a$. Since the Y_j are independent, the probability density of the compound variable Z_α is the convolution of the densities of the compound variables, i.e.

$$f_Z(z) = \begin{cases} c_1\, c_2\ (f_{Y_1} \star f_{Y_2})(z) & z > 0 \\ c_0\ f_{Y_0}(z) & z \leq 0. \end{cases} \tag{14}$$

Thus, up to a normalization for $|Z| \gg 0$, the distribution is given by

$$f_Z(z) = \begin{cases} \frac{1}{1-3a}\ e^{-\frac{z}{1-2a}} & z \geq 0 \\ \frac{1}{1-a}\ e^{\frac{z}{1-a}} & z \leq 0. \end{cases} \tag{15}$$

Therefore, in a semi-logarithmic plot we see a tent — with an exponential correction for small z — according to

$$\ln f_{Z_\alpha}(z) \;\sim\; \begin{cases} -\frac{z}{1-2a} & z > 0 \\ +\frac{z}{1-a} & z \leq 0. \end{cases}$$

For $\mu = 1$ the distribution is symmetric.

$$\ln f_{Z_1}(z) \;=\; -\ln 2 \;-\; |z|. \tag{16}$$

If $\mu < 1$ $(a > 0)$, positive returns are less probable than in the symmetric case, while if $\mu > 1$, $(a < 0)$, positive

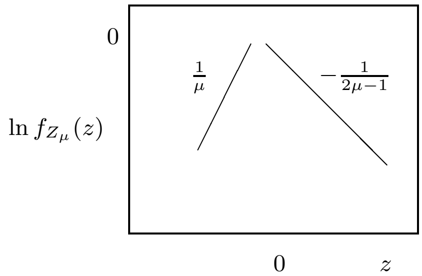

Fig. 6. Positive skewness of the return distribution for $\mu > 1$.

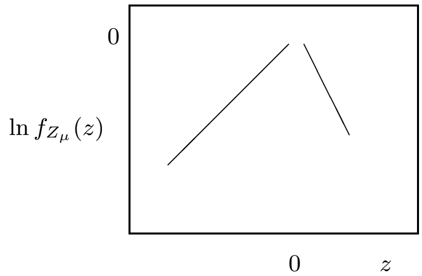

Fig. 7. Negative skewness of the return distribution when $\mu < 1$.

returns are more probable. Hence a positive a, i.e. $\mu < 1$, relates to negative skewness while $\mu > 1$ corresponds to positive skewness. This is in contrast to the case where trading periods all independent, since there the distribution is symmetric for all μ, see [23]

$$\ln f_Z(z) \;=\; -\ln(2\mu) - \frac{|z|}{\mu} \qquad \mu > 0. \tag{17}$$

5 The first-order model: taking all together

We saw in the previous section that using a power function for representing market expectations leads to a Laplacian type of distribution of logarithmic returns. Comparison with real data showed that the agreement is not bad, while there are two systematic deviations. Small returns are less probable than in the exponential distribution while large returns are more likely. Both effects were identified to be due to the following: If demand decreases with increasing prices, small returns become less probable. Further, if the market expectation fluctuates, large returns become more likely. Further, when considering dynamics, our model exhibits an asymmetric distribution. Finally let us therefore consider the 'full-blown' model proposed. The assumptions made are the following

- A *The structure of the market: one asset with stationary and uniformly distributed uncertain value.*
- B_2 *The structure of trading periods: consecutive periods are dependent.*
- C_2 *The 'market expectation': believes about the future growth rate fluctuate over time according to*

$$\varphi(r) = r^{\mu_t}.$$

Figure 8 displays the stylized facts generated by this model. Inspections show that the return trail exhibits '*volatility clustering*', measured by the slow (polynomial) decay of the autocorrelation of squared returns. The generalized Hurst spectrum is non-linear, while the distribution is slightly skewed and exhibits tails that are more heavy than those of an exponential distribution. In summary, this model generates return trails whose statistical properties are in good agreement with respective stylized facts from empirical asset returns.

6 On 'the' entropy of our stylized financial market

Entropy can be regarded as a measure of the *unpredictability of the market*: the higher the entropy is the less predictable the time series of returns is. Furthermore, while even an infinite sequence of moments does generally not determine a distribution uniquely [24], entropy gives an additional characterization of a distribution which extends the consideration of its moments. The entity we consider is the empirical 'entropy' of the real financial market compared with the entropy given by our model. The comparison of both will give us some hints concerning the goodness of the zero-order approximation.

Concerning entropy the following has to be kept in mind: without a deeper understanding of the microscopic basis, no unique choice between different 'entropies' is possible. While the Boltzmann-Gibbs entropy is adequate in the regime of strong chaos (exponential mixing), many natural system including economic system do not accommodate this hypothesis. This should be kept in mind concerning recent approaches to use entropy related principles for option pricing [25]. Apart from the Boltzmann-Gibbs entropy, two alternative entropy measures have been considered in the past. One is the *Renyi entropy* which has turned out to be useful for characterizing multi-fractal systems, the other one is the *Tsallis entropy* [26] which has been shown to be interesting particularly for systems which exhibit global correlations. Since we can not exclude that multi-fractal systems can provide at least a good approximate description of a financial market, one should take the Renyi entropy into account. On the other hand, due to the existence of long-range correlations, a financial market might be considered as a non-extensive system, see [27,28] for considerations about the value of the non-extensive statistical approach in finance. Therefore the consideration of the Tsallis entropy is also in place.

Consequently we will look at the outcome of our model through two glasses: one is the Renyi entropy, the other one is the Tsallis entropy. We use these two measures as instruments to judge about to which extent our model reproduces corresponding properties of empirical data either in one or in the other framework. For this purpose we calculate the Renyi and the Tsallis entropies in our model and compare it to empirical data.

Our zero-oder model obeying assumption A, B_1, C_1 with φ a constant power function model implies that returns are (double) exponentially distributed with some

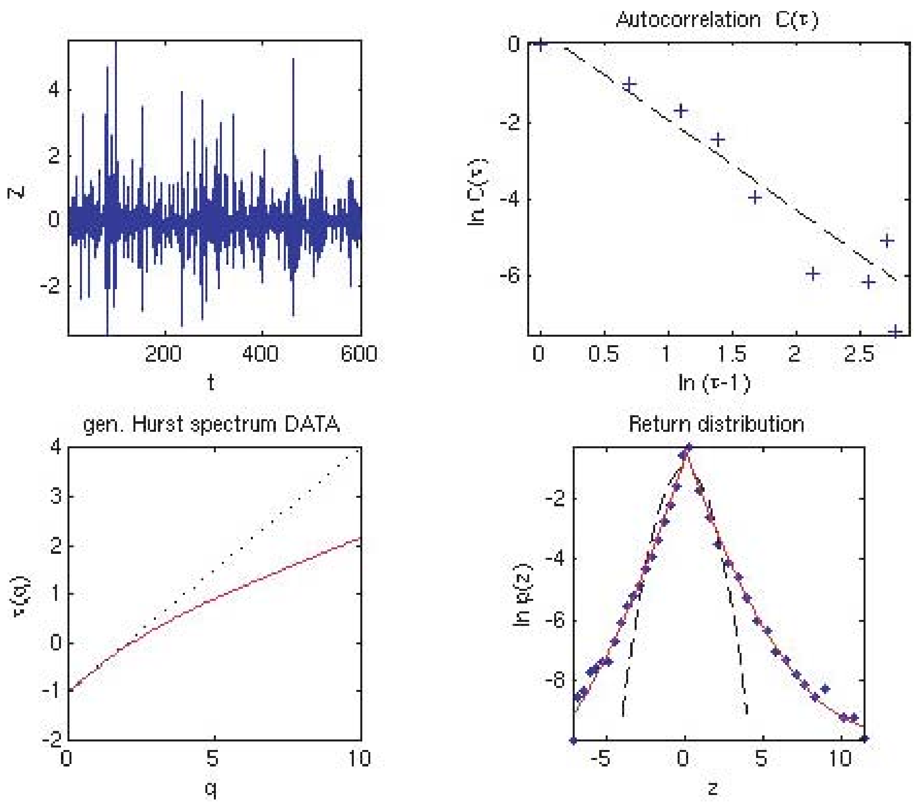

Fig. 8. Stylized facts generated by our model, the return trail, volatility clustering, multi scaling spectrum, and the return distribution.

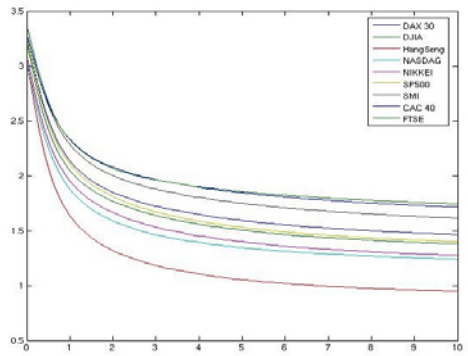

Fig. 9. The Renyi entropies R_β of daily returns of the indices considered.

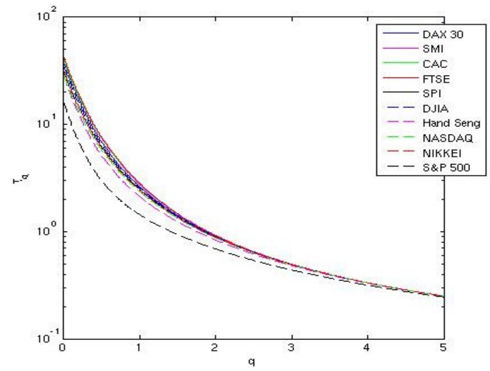

Fig. 10. The Tsallis entropies T_β of daily returns of the indices considered.

non-negative parameter μ. We therefore estimate the related Renyi entropy defined by

$$R_\beta(p) \;=\; \frac{1}{1-\beta}\;\ln\sum_{i=1}^{r}\,p_i^\beta, \quad \beta = 1, 2, ... \tag{18}$$

as well as the Tsallis entropy

$$T_q(p) \;=\; \frac{1}{q-1}\;\left(1-\sum_{k=1}^{r}\pi_k^q\right), \quad q = 1, 2, ... \tag{19}$$

for our model, this gives R_β^{th} and T_q^{th}, and compare each with the respective entropy estimated directly from the data, i.e. R_β^{emp} and T_q^{emp}. We consider a trail of log-returns Z and partition the range into $2r > 0$ cells each of length $1/r$. For later purposes we define

$$C_\beta \;:=\; \frac{1}{2}\;\frac{e^{\frac{\beta}{\mu\, r}}-1}{1-e^{-\frac{\beta}{\mu}}} \tag{20}$$

Note that $\lim_{\beta\to 0} C_\beta(p) = \frac{1}{2\,r}$. Normalized probabilities for micro-states $k = -r+1\ldots r$ are

$$\pi_k \;=\; \begin{cases} C_\beta\; e^{\frac{k-1}{\mu\, r}} & -r \le k \le 0 \\ C_\beta\; e^{-\frac{k}{\mu\, r}} & 1 \le k \le r. \end{cases} \tag{21}$$

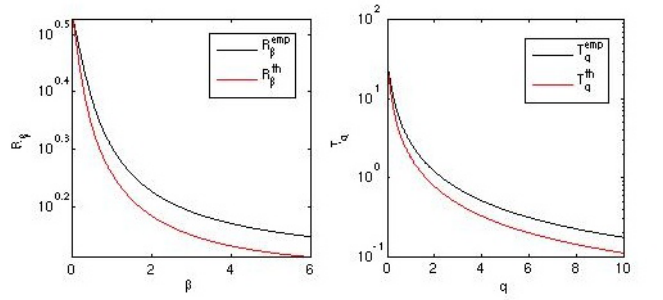

Fig. 11. The Renyi entropy R_β and the Tsallis entropy T_q for the NIKKEI 225. The theoretical entropies (dashed lines) are calculated under the hypothesis that log returns are strictly Laplacian distributed.

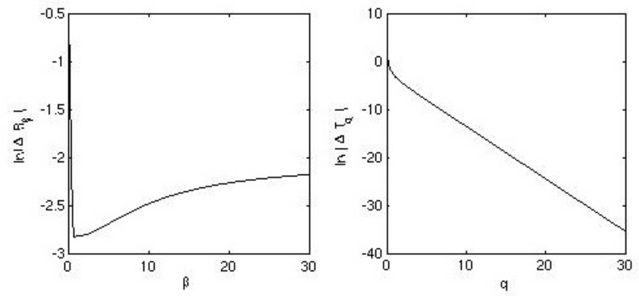

Fig. 12. Comparison between the theoretical and the empirical entropies.

The Renyi entropy and the Tsallis entropy then yield

$$R_\beta(p) \;=\; \frac{1}{1-\beta}\;\ln\,\frac{C_1^\beta}{C_\beta} \tag{22}$$

while the Tsallis entropy reads

$$T_q(p) \;=\; \frac{1}{q-1}\;\left(1-\frac{C_1^q}{C_q}\right). \tag{23}$$

Note that $\lim_{\beta\to 0} R_\beta(p) = \ln\, 2\, r$, while $\lim_{q\to 0} T_q(p) = 2\,r-1$, where $2\,r$ is the number of cells considered.

We calculate the Renyi entropy R_β as well as the Tsallis entropy T_q for the indices listed in Table 1 for a fixed number of cells $r = 30$. In the following we restrict ourselves to the NIKKEI 225 see Figure 11. Former estimates of the asymmetry of the distribution of negative returns and positive returns, measured by the parameters $\mu_\pm$ showed that the distribution is (almost) symmetric. For the NIKKEI 225 considered we have for the distribution of positive and negative returns, respectively:

$$\mu_+ \;=\; 0.728 \qquad \mu_- \;=\; 0.742.$$

Hence we considered the trail of absolute returns, i.e. $|\ln Z|$. We normalized returns to the unit interval by considering $|Z^*| = \frac{|Z|_{emp}}{\max|Z|_{emp}}$ and estimated the corresponding parameter μ from fitting the returns to a Laplace distribution giving $\mu = 0.776$. This parameter then is taken for calculating the theoretical entropies according to Table 3. Corresponding graphs are the dashed lines, while the solid lines are the empirical entropies from the data. Deviation between the empirical entropies and the respective theoretical entropies are seen to be fairly small.
Figure 12 displays these differences as a function of β and q respectively. Here

$$\Delta R_\beta = R_\beta^{emp} - R_\beta^{th}, \qquad \Delta T_q = T_q^{emp} - T_q^{th}$$

While ΔR_β is asymptotically constant and positive, ΔT_q asymptotically decays exponentially in q.

These results has given support to our working hypothesis that the market expectation function is well approximated by a power function $\varphi(r) = r^\mu$ with some positive parameter $\mu > 0$ for the following estimations. Here have to keep in mind, that the empirical entropy, whether it is the Renyi or the Tsallis, is always larger then the entropy, which was calculated based on the assumption that the distribution is strictly Laplacian. This means that the empirical distribution is broader then the theoretical one, which is also seen in Figure 1: The Laplacian does not have enough mass in its tails!

6.1 Fluctuating market expectations and the effect on entropies

Assumptions are A, B_1, C_2, *i.e.* $\varphi_t(\lambda) = \lambda^{\mu_t}$, *where* μ_t *fluctuates uniformly in time.*

Since fluctuations increase the broadness of the return distribution, one expects that the corresponding entropies are increasing in the fluctuating parameter. This is indeed justified by the following estimation. For the sake of simplicity we consider the case of an infinite number of cells, in which the Renyi and the Tsallis entropy become

$$R_\beta(\tilde f_Z) = \frac{1}{1-\beta}\;\ln\int_{\mathbb{R}} \tilde f_Z^\beta\, dz$$
$$T_q(\tilde f_Z) = \frac{1}{q-1}\left(1-\int_{\mathbb{R}} \tilde f_Z^q\, dz\right)$$

Expanding the solution for small ϵ we obtain

$$T_q(\tilde f_Z) = \frac{q-(2\,\mu)^{1-q}}{q\,(q-1)} + \frac{\epsilon^2}{3\,\mu}\,\frac{(1-q)2^{1-q}}{q^2\,\mu^q} + \mathcal{O}(\epsilon^4) \tag{24}$$

$$R_\beta(\tilde f_Z) = \frac{1}{1-\beta}\ln\frac{(2\,\mu)^{1-\beta}}{\beta} + \frac{\epsilon^2}{3\,\mu^2}\,\frac{1-\beta}{\beta} + \mathcal{O}(\beta^4). \tag{25}$$

For notational convenience we write $T_q^0(\tilde f_Z)$ and $R_\beta^0(\tilde f_Z)$ for the Tsallis and the Renyi entropies with $\epsilon = 0$ respectively. Further we define $\Delta T_q(\tilde f_Z) := \left|\, T_q(\tilde f_Z) - T_q^0(\tilde f_Z)\,\right|$

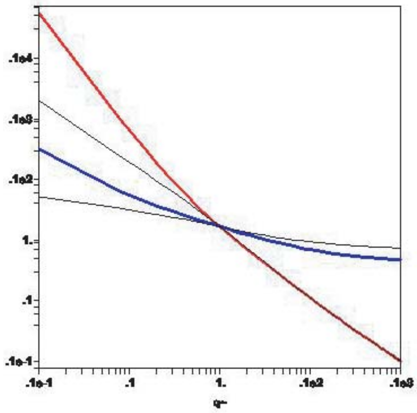

Fig. 13. Entropies $T_q(\tilde{f}_Z)$ (red) and $R_\beta(\tilde{f}_Z)$ are shown as compared to the entropies for constant market expectations (thin black lines).

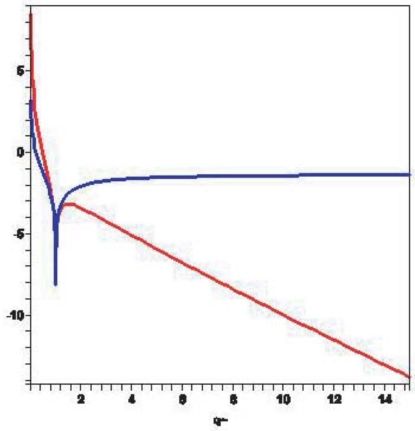

Fig. 14. For large β, q, $\Delta R_\beta(\tilde{f}_Z) < 0$, while $\Delta T_q(\tilde{f}_Z) \sim 0$.

and $\Delta R_\beta(\tilde{f}_Z) := \left| R_\beta(\tilde{f}_Z) - R^0_\beta(\tilde{f}_Z) \right|$, respectively. Obviously

$$\lim_{q\to\infty} \Delta T_q(\tilde{f}_z) = 0$$

$$\lim_{\beta\to\infty} \Delta R_\beta(\tilde{f}_z) = -\frac{1}{3}\left(\frac{\epsilon}{\mu}\right)^2 \;<\; 0.$$

Therefore, the effect of fluctuations vanishes for the Tsallis entropy with increasing q, while is does not for the Renyi entropy.

Observation 2 *Fluctuations of the market expectations increase entropy for $\beta, q < 1$. corresponding to creating more mass in the tails.*

This means that the entropy increases when taking fluctuations into account. Recall that in the case of our model with constant parameter μ, the theoretical entropy was less than the empirically observed one. From this part of investigation we learn that an important ingredient seems to be that market expectations are not constant but fluctuate in time, which again seems to be economically plausible. This finding can be seen as related to D. Farmers suggestions that large fluctuations are caused by liquidity fluctuations [29].

7 Conclusion

All models are wrong but some are useful.
G.E.P. Box, 1979

The existence of stylized facts suggests that price trails of different financial markets might be regarded as different realizations of a more general stochastic system, called 'The' financial market. If so then the question is about the nature of this system, i.e. whether there exists a unique mechanism driving price formation which is common to 'all' financial markets. The model proposed is kept as simple as possible to allow for successive generalizations to, hopefully, approach the situation on a real financial market. The model is based on the following idea: Since a price is a macro-observable of a financial market, the model about price dynamics is defined on the macro level, while any attempt to model the macro level analogous to some micro level including 'micro foundation' is avoided. Since we are dealing with the prices trail of indices, the model is 1-dimensional. A fairly strong assumption is that distributions of driving random variables are stationary and uniformly distributed in some finite interval. While, in principle respective considerations can be conducted for other distributions, we chose the uniform distribution to insert a minimal amount of additional information additional to what can be strictly observed.

"*Price formation is endogenous in the process of trading and driven by the expectations of investors about future growth rates of the value of an asset*". This establishes a multiplicative stochastic negative feedback in the dynamics of price formation. While on very long time scales, market expectations might be well approximated by their (time constant) mean, on shorter time scales it is reasonable to assume that 'the market expectations' fluctuates in time. This model is simple enough to be analyzed in some detail which open the door for further considerations and specifications.

The corresponding price dynamics model indeed exhibits statistical properties which are in good agreement with so-called 'stylized facts' drawn from empirical data, including 'volatility clustering', measured by the slow decay of autocorrelations of integer powers of returns and a non-trivial spectrum of generalized Hurst exponents which characterizes the irregularity of the return trail considered. The return distribution in this model shows a slight skewness and, in the presence of fluctuations in the market expectation, tails are more heavy than those from a Laplacian distribution. Particularly tails of this distribution decay exponentially for large returns, which is well-known from empirical returns in high-frequency data.

Taking all this together, the mechanism represented by our model seems to be a reasonable candidate to drive

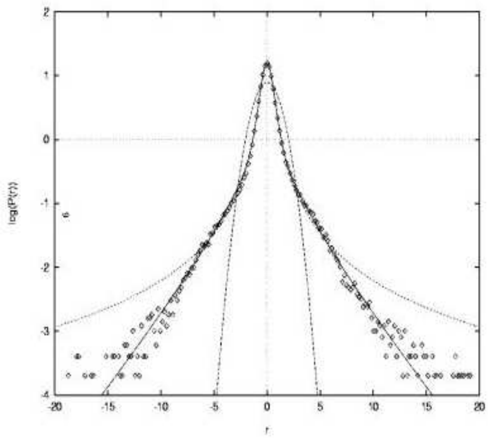

Fig. 15. Return distribution of the S&P 500, high frequency data, adapted from [8].

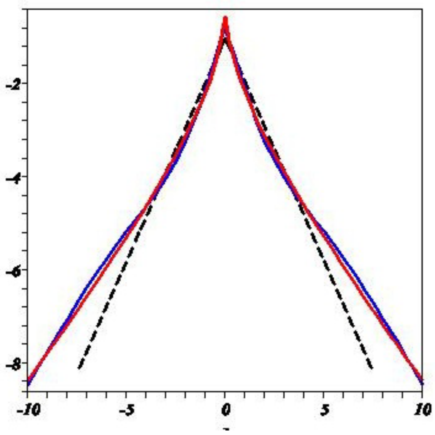

Fig. 16. Return distribution in our model with fluctuating market expectations, see Figure 4.

prices on financial markets in general. While this is good news on the one hand, it also shows something else: Not too much is needed to reproduce major stylized facts. Hence stylized facts, as apparent in the literature are not sufficient to single out a unique sound economic model. More data about regularities of prices (returns) and other factors moving the market such as fluctuations in liquidity are needed to approach an understanding of financial markets' dynamics. This, in my opinion, concerns particularly the necessity to consider the dynamics of market micro structure.

One could draw the following conclusion: while The Financial Market might be simple to understand concerning its statistical properties, the real world problem consists in trading on single realizations. Considerations of the 'entropy' of the financial market, may serve as a hint to which extent statements about the predictability of the financial market are possible.

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